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ON THE SECOND EIGENVALUE OF THE DIRICHLET LAPLACIAN

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ABSTRACT

The multiplicity of the second eigenvalue of the Dirichlet Laplacian on smooth Riemannian surfaces with boundary that satisfy certain convexity condition is at most two. The proof is based on variational formulas for eigenvalues under the change of the domain.

1. Introduction

Let Ω be a simply connected bounded submanifold with smooth boundary of an open Riemann surface M. We are interested in eigenfunctions of the Laplacian in Ω with the Dirichlet's boundary conditions:

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega; \\ u = 0 & \text{in } \partial\Omega. \end{cases} \tag{1.1}$$

It is well known that the spectrum of the problem (1.1) is discreet and positive, and the smallest eigenvalue λ_1 is simple. The second eigenvalue λ_2 might be multiple (and it is in fact multiple in the case of a circle). C.-S. Lin proved in [1] that for convex domains in R^2 the multiplicity of λ_2 is at most 2.

Let X be a smooth vector field that is defined in a neighborhood of Ω. I shall assume that X preserves the metric tensor g on M. It means that

$$\mathcal{L}_X g = 0, \tag{1.2}$$

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where $\mathcal{L}_X$ is the Lie derivative. Let ν be the field of unit outward normal vectors to $\partial\Omega$. Let

$$\Gamma_X^{\pm} = \{x \in \partial\Omega\colon\ \pm(X,\nu)(x) > 0\}.$$

I say that Ω is *X-convex* if both sets $\overline{\Gamma_X^{\pm}}$ are connected, and at least one of them is not empty. I shall prove

THEOREM: *Let a simply connected bounded submanifold with smooth boundary Ω of an open Riemannian surface M is X-convex for a smooth vector field X defined in a neighborhood of Ω that satisfies (1.2). Then the second eigenvalue of the problem (1.1) is of multiplicity at most 2.*

Convex domains in R^2 are X-convex for every constant vector field X. Clearly, constant vector fields preserve the Eucledian metric. However, there are non-convex domains in R^2 that are X-convex for some constant vector field X. The theorem can be easily applied for domains in R^2, on the sphere S^2, and on the hyperbolic plane H^2. In these cases there exist families of vector fields preserving the metric, and the assumption is that Ω is convex with respect at least one of them.

The basic tool of proving theorems of such a type is investigating nodal lines for the solutions of the problem (1.1). One can find basic fact about nodal lines in [1,2]. If u is an eigenfunction, then the nodal line N_u is the closure of the set $\{(x,y) \in \Omega : u(x,y) = 0\}$. The Courant nodal domain theorem [3] says that the nodal line N_u divides the domain Ω into at most k connected components, where k is the number of the corresponding eigenvalue. In the case $k = 2$ the number of subdomains equals exactly 2. From this point I shall discuss only the case $k = 2$. There are two possible cases:

(i) $N_u \cap \partial\Omega \neq \emptyset$;

(ii) $N_u \cap \partial\Omega = \emptyset$. To prove the theorem, it is sufficient to rule out the second possibility (e.g., see [1]). I shall show it for the sake of completeness.

First, I would like to describe the method of the proof. Assume that $N_u \cap \partial\Omega = \emptyset$. The nodal line N_u divides our domain Ω into an internal part Ω_+ and an external part Ω_-. Let us move domain Ω along the vector field X. By $\Omega(t)$, $-\epsilon < t < \epsilon$, I denote the image $\Phi(t)\Omega$, where $\Phi(t)$ is the one-parameter group of local isometries generated by X. Let $\lambda^-(t)$ be the first eigenvalue of the Dirichlet Laplacian in $\Omega_-(t) = \Omega(t) - \Omega_+$. I claim that

$$\lambda^-(t) \geq \lambda_2. \tag{1.3}$$

Let ψ_+ be the first eigenfunction of the Dirichlet Laplacian in Ω_+, extended by 0 in $\Omega_-(t)$, and let ψ_- be the first eigenfunction of the Dirichlet Laplacian in $\Omega_-(t)$, extended by 0 in Ω_+. Let $u(x)$ be a linear combination of ψ_+ and ψ_-, orthogonal to the first eigenfunction of the Dirichlet Laplacian in $\Omega(t)$. If the inequality (1.3) does not hold then

$$\int_{\Omega(t)} |\nabla u|^2 dx < \lambda_2 \int_{\Omega(t)} u^2 dx,$$

and this contradicts to the min-max principle (note that Ω and $\Omega(t)$ are isospectral). The inequality (1.3) implies

$$\frac{\partial \lambda^-}{\partial t}(0) = 0. \tag{1.4}$$

In section 2 I shall derive formulas for the first and for the second variations of the first eigenvalue of the Dirichlet Laplacian under the change of the domain. The formula for the first variation is rather well-known folklore result. In the case of an analytic deformation the proof can be found in [4]. In the section 3 I shall complete the proof of the theorem.

2. Variations of the first eigenvalue of the Dirichlet Laplacian

Let M be a smooth open Riemannian manifold of any dimension, and let G be a closed submanifold of M with smooth boundary. The dimensions of M and G are supposed to be equal. Let V be a smooth vector field, defined in the neighborhood of the boundary ∂G of the manifold G, and let ν be the field of unit outward normal vectors to ∂G. For small values of a real variable t manifolds with boundary $G(t)$ are defined. These manifolds are bounded by images of ∂G under the flow Φ_t generated by the vector field V. Let $\lambda(t)$ be the first eigenvalue of the Dirichlet Laplacian in $G(t)$.

LEMMA 2.1: *Under the above assumptions,*

$$\lambda'(0) = -\int_{\partial G} \left(\frac{\partial \psi}{\partial \nu}\right)^2 (V, \nu) dS, \tag{2.1}$$

where $\psi(x)$ is normalized first eigenfunction of the Dirichlet Laplacian in G, $(\cdot,\cdot)$ is the Riemannian scalar product, and dS is the canonical measure of the induced Riemannian structure on ∂G.

Proof: Let us fix notations. We denote by $g_\#$ the metric tensor on M (with lower indices), by $g^\#$ we denote the metric tensor with upper indices, and g is, as usual, the determinant of the matrix $g_\#$. Let T be a small tubular neighborhood of ∂G in M. One can introduce coordinates (s, x') in T, where s is the distance from the point to ∂G, and x' are local coordinates on ∂G; the coordinates x' are constant along geodesics normal to ∂G. In these coordinates the metric tensors $g_\#$ and $g^\#$ are of the form

$$g_\# = \begin{pmatrix} 1 & 0 \\ 0 & g_\#(s,x') \end{pmatrix}, \quad g^\# = \begin{pmatrix} 1 & 0 \\ 0 & g^\#(s,x') \end{pmatrix}.$$

By $p(x')$ we shall denote the scalar product $(V(0,x'), \nu(x'))$.

We shall keep in all computations terms up to the first degree in t. We write $f(t) \sim g(t)$ if $f(t) - g(t) = O(t^2)$. In the coordinates (s, x') the boundary $\partial G(t)$ of $G(t)$ is given by

$$s = w(x', t), \quad \text{where} \quad w(x', t) \sim tp(x'). \tag{2.2}$$

The manifold $G(t)$ is described near $\partial G(t)$ by the inequality $s < w(x', t)$. Now I am going to introduce a diffeomorphism $F_t^\sigma \colon G(t) \to G$. Let σ be a small number, and let $\alpha(\tau, \sigma, w)$ be a smooth function of three real variables $\sigma > 0$, $w > -\sigma$, and $\tau \leq w$, such that it increases in τ, and $\alpha(\tau, \sigma, w) = \tau$ when $\tau \leq -\sigma$. The mapping F_t^σ is identical out of the region $U_\sigma = \{s > -\sigma\}$, and

$$F_t^\sigma(s, x') = (\alpha(s, \sigma, w(x', t)), x') \tag{2.3}$$

in U_σ. Let $g_\#(t, \sigma)$ be the push forward of the metric $g_\#$ by the diffeomorphism F_t^σ. The metrics $g_\#$ and $g_\#(t,\sigma)$ coincide out of U_σ. The first eigenvalue of the Dirichlet Laplacian $\Delta(t)$ in G that corresponds to the metric $g_\#(t)$ equals $\lambda(t)$. Let $\psi(t)$ be the first eigenfunction of $\Delta(t)$. Let $g(t) = \det g_\#(t)$. Operators $(g(t)/g)^{1/4}\Delta(t)(g(t)/g)^{-1/4}$ form a smooth family of selfadjoint operators with compact resolvent. Therefore, $\lambda(t)$ is a smooth function of t [5]. One has

$$\Delta(t)\psi(t) + \lambda(t)\psi(t) = 0,$$

and

$$\Delta'(0)\psi + \Delta\psi'(0) + \lambda'(0)\psi + \lambda\psi'(0) = 0.$$

One can take the scalar product (which is induced by the metric $g_\#$) of the both parts of the last equality with ψ. After integrating by parts one gets the usual Rayleigh's formula

$$\lambda'(0) = -(\Delta'(0)\psi, \psi). \tag{2.4}$$

I need a different representation for $\lambda'(0)$. One has

$$\begin{aligned}(\Delta(t)\psi,\psi) &= \int_G \Delta(t)\psi\cdot\psi\sqrt{\frac{g}{g(t)}}\sqrt{g(t)}dx\\ &= -\int_G |\nabla\psi|_t^2\sqrt{g}dx - \int_G \psi\left(\nabla\psi,\nabla\sqrt{\frac{g}{g(t)}}\right)_t\sqrt{g(t)}dx\\ &= -\int_G |\nabla\psi|_t^2\sqrt{g}dx - \frac{1}{2}\int_G \psi(\nabla\psi,\nabla\kappa)_t\sqrt{g}dx,\end{aligned}$$

where $\kappa = \log(g(t)/g)$, and

$$(\nabla\psi,\nabla\kappa)_t = g^{ij}(t)(\partial\psi/\partial x^i)(\partial\kappa/\partial x^j).$$

Finally,

$$\lambda'(0) = \frac{d}{dt}\left[\int_G |\nabla\psi|_t^2\sqrt{g}dx + \frac{1}{2}\int_G \psi(\nabla\psi,\nabla\kappa)_t\sqrt{g}dx\right]_{t=0}. \tag{2.5}$$

The value of $\lambda'(0)$ depends neither on the choice of the function α nor on the choice of the number σ. To perform all computations it is convenient to make a special choice of the function α, namely,

$$\alpha(\tau,\sigma,w) = \begin{cases}\tau, & \text{if } \tau \le -\sigma;\\ \sigma(\tau-w)/(\sigma+w), & \text{if } \tau \ge -\sigma.\end{cases} \tag{2.6}$$

The only problem is, this function is not smooth, and the corresponding metric $g_{\#}(t)$ is not continuous. Nevertheless, the formula (2.5) holds if all derivatives are treated in the sense of distributions. Rigorously speaking, one can take a sequence of smooth functions α_n converging to the function α in the sense of distributions, and then one can take the limit in (2.5).

Now I am going to compute components of the metric tensor $g^{\#}(t)$ in the strip $V_\sigma = \{-\sigma \le s \le 0\}$. One has

$$s = \sigma\frac{\tau - tp}{\sigma + tp} \sim \tau - tp\left(1+\frac{\tau}{\sigma}\right)$$

and

$$\tau \sim (s+tp)\left(1-\frac{t}{\sigma}p\right)^{-1} \sim s + tp\left(1+\frac{s}{\sigma}\right).$$

Hence,

$$ds \sim \left(1-\frac{t}{\sigma}p\right)d\tau - t\left(1+\frac{\tau}{\sigma}\right)dp \sim \left(1-\frac{t}{\sigma}p\right)d\tau - t\left(1+\frac{s}{\sigma}\right)dp$$

because $s-\tau=O(\tau)$. Components of metric tensor $g^{\#}(t)$ equal

$$g^{ss}(t)\sim\left(1-\frac{t}{\sigma}p\right)^2\sim 1-\frac{2t}{\sigma}p,$$
$$g^{s\beta}\sim -t\left(1+\frac{s}{\sigma}\right)g^{\beta\gamma}(s,x')\frac{\partial p^{\gamma}}{\partial x},\quad\text{and}$$
$$g^{\beta\gamma}=g^{\beta\gamma}(\tau,x')\sim g^{\beta\gamma}(s,x')+tp\left(1+\frac{s}{\sigma}\right)\frac{\partial g^{\beta\gamma}(s,x')}{\partial s}.$$

I use the standard convention: summation over repeated indices is always assumed. The Greek letters are being used to denote tangential coordinates x'. The last piece of information we need to apply (2.5), is

$$\kappa=-\log\left(1-\frac{2t}{\sigma}p\right)-\log\det\left(g^{\#}+tp\left(1+\frac{s}{p}\right)\frac{\partial g^{\#}}{\partial s}\right)+\log\det g^{\#}$$
$$\sim\frac{2t}{\sigma}p-tp\left(1+\frac{s}{\sigma}\right)\operatorname{tr}\frac{\partial g^{\#}}{\partial s}g^{\#}.$$

The first term in the right hand side of (2.5) is equivalent to

$$\int_G|\nabla\psi|_t^2\sqrt{g}dx\sim\int_G|\nabla\psi|^2\sqrt{g}dx-\frac{2t}{\sigma}\int_{V_\sigma}p(x')\psi_s^2\sqrt{g}dx$$
$$-t\int_{V_\sigma}g^{\beta\gamma}\left(1+\frac{s}{\sigma}\right)p_\gamma\psi_s\psi_\beta\sqrt{g}dx+t\int_{V_\sigma}p\left(1+\frac{s}{\sigma}\right)\frac{\partial g^{\beta\gamma}}{\partial s}\psi_\beta\psi_\gamma\sqrt{g}dx.$$

Here ψ_s is the s-derivative, and ψ_γ is x^γ-derivative. The second term in the right hand side of (2.5) is equivalent to

$$\frac{t}{\sigma}\int_{\partial G}p(x')\sqrt{g(-\sigma,x')}\psi(-\sigma,x')\psi_s(-\sigma,x')dx'+t\int_{V_\sigma}\psi(\nabla\psi,\nabla\omega)\sqrt{g}dx,$$

where

$$\omega=\frac{p}{\sigma}-\frac{p}{2}\left(1+\frac{s}{\sigma}\right)\operatorname{tr}g_s^{\#}g_{\#}.$$

Therefore,

$$\lambda'(0)=-\frac{2}{\sigma}\int_{V_\sigma}p(x')\psi_s^2\sqrt{g}dx+\frac{1}{\sigma}\int_{\partial G}p(x')\psi(-\sigma,x')\psi_s(-\sigma,x')\sqrt{g}dx'$$
$$-\int_{V_\sigma}g^{\beta\gamma}\left(1+\frac{s}{\sigma}\right)p_\gamma\psi_s\psi_\beta\sqrt{g}dx+\int_{V_\sigma}p\left(1+\frac{s}{\sigma}\right)\frac{\partial g^{\beta\gamma}}{\partial s}\psi_\beta\psi_\gamma\sqrt{g}dx$$
$$+\int_{V_\sigma}\psi(\nabla\psi,\nabla\omega)\sqrt{g}dx.$$

The right hand side of the last formula does not depend on σ. Limits of the last three terms equal 0 when $\sigma \to 0$. The first and the second terms have the limits

$$-2\int_{\partial G} p(x')\sqrt{g}dx' \quad \text{and} \quad \int_{\partial G} p(x')\sqrt{g}dx'$$

respectively. Finally,

$$\lambda'(0) = -\int_{\partial G} p(x')\sqrt{g}dx'. \qquad \blacksquare$$

Now I am going to derive a formula for the second derivative of λ. From this point $\dim M = 2$. One can rewrite the formula (2.1) in the form

$$\mathcal{L}_V\lambda = -\int_{\partial G}\left(\frac{\partial\psi}{\partial\nu}\right)^2 (V,\nu)dS. \tag{2.1'}$$

Let W be another vector field that is defined in a neighborhood of ∂G. Let Ψ^t be the family of local diffeomorphisms associated with W. I shall use coordinates (s, x') in a neighborhood of ∂G (see the proof of Lemma 2.1). The vector field W has the form (W_n, W_τ) in these coordinates. The image Γ_t of ∂G under the mapping Ψ^t is given by the equation

$$s = \beta(x') \sim tW_n(0, x'). \tag{2.7}$$

Clearly, the element of the length along Γ_t equals dS modulo t^2. Therefore,

$$\mathcal{L}_W\mathcal{L}_V\lambda = -\int_{\partial G}(V\psi)\mathcal{L}_W\left(\frac{\partial\psi}{\partial\nu}\right)dS - \int_{\partial G}\left(\frac{\partial\psi}{\partial\nu}\right)\mathcal{L}_W(V,\nu)dS. \tag{2.8}$$

Let $G(t)$ be the domain bounded by Γ_t, let $\psi(t)$ be the normalized first eigenfunction of the Dirichlet Laplacian in $G(t)$, and let $\lambda(t)$ be the corresponding eigenvalue. I denote by $\psi_+(t)$ the function $\psi(t)$, extended by 0 outside $G(t)$. By $'$ I shall denote the t-derivative. One has

$$(\Delta + \lambda)\psi_+ = -\chi\delta_{\Gamma(t)}, \tag{2.9}$$

where $\chi = \partial\psi/\partial\nu$, and $\delta_{\Gamma(t)}$ is the delta-function supported on $\Gamma(t) = \Gamma_t$. Let us equate derivatives of both parts of (2.9) at $t = 0$:

$$(\Delta + \lambda)\mathcal{L}_W\psi_+ + (\mathcal{L}_W\lambda)\psi_+ = -(\mathcal{L}_W\chi)\delta_{\partial G} - \chi(\mathcal{L}_W\delta_{\partial G}). \tag{2.10}$$

I shall take only vector fields W such that

$$\int_{\partial G}\left(\frac{\partial\psi}{\partial\nu}\right)^2(W,\nu)dS=0. \tag{2.11}$$

In this case the second term from the left hand side of (2.10) equals 0. Denote by ω the function $\mathcal{L}_W\chi$. The distribution that appears in the left hand side of (2.10) maps a function f into

$$-\int_{\partial G}\omega f dS-\int_{\partial G}\chi W_n(s)\frac{\partial f}{\partial\nu}dS. \tag{2.12}$$

Therefore, ω is the normal derivative of the solution v of the equation

$$(\Delta+\lambda)v=0$$

with the boundary condition

$$v(0,x')=-W_n(x')\chi=-W\psi.$$

I shall write $\omega=-N(\lambda)(W\psi)$; $N(\lambda)$ is the Neumann operator for $\Delta+\lambda$. This operator is defined only on the subspace H_0 of functions that are orthogonal to $\partial\psi/\partial\nu$. The function $W\psi$ belongs to this subspace because of (2.11). Now we can write the first term from the right hand side of (2.8) in the form

$$2(V\psi,N(\lambda)(W\psi)). \tag{2.13}$$

The next step is to simplify the second term from the right hand side of (2.8). Recall that (W_τ,W_n) are components of W in normal (x',s) coordinates. Let (V_τ,V_n) be components of the vector field V. The outward normal vector to the curve Γ_t is of the form

$$\nu_t\sim\frac{\partial}{\partial s}-t\frac{\partial W_n(x')}{\partial x'}\frac{\partial}{\partial x'}$$

(see (2.7)). Therefore,

$$(\nu_t,V(x',tW_n(x')))\sim V_n(0,x')+t\left(\frac{\partial V_n}{\partial s}(0,x')W_n(0,x')-V_\tau(0,x')\frac{\partial W_n}{\partial x'}(0,x')\right).$$

Hence the second term from the right hand side of (2.8) equals

$$\Xi(V,W)=-\int_{\partial G}\left(\frac{\partial\psi}{\partial\nu}\right)^2\left(\frac{\partial V_n}{\partial\nu}W_n-V_\tau\frac{\partial W_n}{\partial x'}\right)dS \tag{2.14}$$

We have proved

LEMMA 2.2: *Let V and W be vector fields defined in a neighborhood of ∂G, and let they satisfy (2.11). Then*

$$\mathcal{L}_W \mathcal{L}_V \lambda = 2(N(\lambda)(W\psi), V\psi) + \Xi(V,W), \tag{2.15}$$

where $\Xi(V,W)$ is defined by (2.14).

3. Proof of the theorem

Let a domain Ω and a vector field X satisfy the assumptions of the theorem. Assume that the dimension of the eigenspace M of the Dirichlet Laplacian in Ω that corresponds to the second eigenvalue is at least 3. Then there exists $\psi \in M$ such that

$$N_\psi \cap \partial\Omega = \emptyset. \tag{3.1}$$

In fact, if (3.1) is not satisfied for any function $\psi \in M$, $\psi \neq 0$, then sets

$$S_\psi = \{x \in \partial\Omega : \partial\psi/\partial\nu \geq 0\}$$

are nonempty closed intervals (S_ψ may be equal to a point). This easily follows from the Courant's nodal line theorem (e.g., see [1]). We identify the boundary $\partial\Omega$ with the standard circle S^1. Fix a point $P \in \partial\Omega$. Then any point $Q \in \partial\Omega$ corresponds to $\exp(2\pi i|PQ|/|\partial\Omega|)$ where $|PQ|$ is the distance from P to Q in the counterclockwise direction, and $|\partial\Omega|$ is the length of $\partial\Omega$. One can assign the center of S_ψ to any non-zero function $\psi \in M$. This construction gives rise to a continuous odd function $M - 0 \to S^1$. The Borsuk–Ulam theorem yields $\dim M \leq 2$.

Let $u \in M$, and $\psi(t) = \psi + tu$. For small values of $|t|$ one has

$$N_{\psi(t)} \cap \partial\Omega = \emptyset.$$

Now one can apply Lemma 2.1 to the domain Ω_- corresponding to $\psi(t)$, and to a vector field that equals X in a neighborhood of $\partial\Omega$, and that equals 0 in a neighborhood of $N_{\psi(t)}$. Formulas (1.4) and (2.1) imply

$$\int_{\partial\Omega} \left(\frac{\partial\psi}{\partial\nu} + t\frac{\partial u}{\partial\nu}\right)^2 (X,\nu)dS = 0 \tag{3.2}$$

for all sufficiently small t. Therefore,

$$\int_{\partial\Omega} \frac{\partial\psi}{\partial\nu}\frac{\partial u}{\partial\nu}(X,\nu) = 0. \tag{3.3}$$

The assumption of X-convexity of Ω yields the existence of two points $P, Q \in \partial\Omega$ such that (X,ν) does not change the sign on both arcs connecting P and Q. Note that the signs of (X,ν) on these arcs are different. In fact, if the scalar product (X,ν) does not change the sign on $\partial\Omega$ then $(X,\nu) \equiv 0$ because of the formula (3.2) with $t = 0$ ($\partial\psi/\partial\nu \neq 0$). If $\dim M \geq 3$, one can find a function $u \in M$ with

$$\frac{\partial u}{\partial\nu}(P) = \frac{\partial u}{\partial\nu}(Q) = 0,$$

and the product $(X,\nu)\partial u/\partial\nu$ has the constant sign on $\partial\Omega$ (see Lemma 1.2 in [1]). Therefore, for this choice of the function u the integrand in (3.3) does not change the sign, and it does not equal 0 identically (see Lemma 1.3 in [1]). This contradiction completes the proof. ∎

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